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ESTIMATES ON THE DISTANCE OF INERTIAL MANIFOLDS

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Abstract. In this paper we obtain estimates on the distance of the inertial manifolds for dynamical systems generated by evolutionary parabolic type equations. We consider the situation where the systems are defined in different phase space and we estimate the distance in terms of the distance of the resolvent operators of the corresponding elliptic operators and the distance of the nonlinearities of the equations.

1. Introduction

Many systems coming from Partial Differential Equations of evolutionary type, enjoy the property of having a finite dimensional manifold which is smooth, invariant and exponentially attractive and carries over all the asymptotic dynamic information of the system. All bounded invariant sets (equilibria, periodic orbits, connecting orbits, attractors, etc) lie in this invariant manifold. The existence of these manifolds is proved once we guarantee that the associated linear elliptic operator of the system has large enough gaps in the spectrum and it is obtained through an appropriate fixed point argument. Proving that we have these gaps is one of the major difficulties of the theory, but still there is a class of equations (for instance, one dimensional parabolic equations) for which these inertia manifolds exist and once they exist, we can reduce the system to a finite dimensional one, for which more techniques are available. We refer to [4, 16] for general references on the theory of Inertial manifolds. See also [15] for an accessible introduction to the theory. These inertial manifolds are smooth, see [7]. We also refer to [11, 9, 3, 16, 5, 8] for general references on dynamics of evolutionary equations.

Just because of the relevance of these manifolds, it is very important the analysis of its behavior under perturbations of the equation. Identifying the kind of perturbations allowed so that the inertial manifold persists and estimating the distance of the inertial manifolds is an important task which have implications in the analysis of the dynamics of the equations. One of the first examples in which an analysis of the persistence of inertial manifolds was carried over was in [10], where the dynamics of a parabolic equation in a thin domain is analyzed. This paper has been one of the main motivations for our work. In the case treated in [10], the limit equation is one-dimensional for which the gap condition is satisfied since the elliptic operator is of Surm-Liouville type and spectral gaps are known to exist. The inertial manifold of the limiting one-dimensional problem is proved and after an analysis of the continuity of the spectrum under this perturbation, the inertial manifold is lifted to the perturbed 2-dimensional problem in the thin domain. An estimate of the distance of the inertial manifolds is provided, although it is not as sharp as the one we obtain in this paper. Also, some general results on persistence can be found in [4], and also in [12], where the results are more focused on the numerical approximations of the equations. More recently some results on the behavior of these manifolds under perturbation of the domain have appeared [13, 17], although they do not provide estimates on the distance of the manifolds.

In this work we provide estimates on the distance between the inertial manifold of a system and the inertial manifold of a perturbation of it. The systems may have different phase space (so we may apply these techniques to domain perturbation problems) and the distance is estimated in terms of two parameters

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only: the distance of the resolvent operators of the elliptic part and the distance of the nonlinearities of the equations, see Theorem 2.2.

We describe now the contents of the paper.

In Section 2 we introduce the notation, the main hypothesis that we will impose, **(H1)** related to the convergence of the resolvent operators and **(H2)** related to the convergence of the nonlinarities. We also state the main result of the paper, Theorem 2.2.

In Section 3 we analyze the behavior of the linear part of the equations. We show the convergence of the spectrum once the resolvent convergence is imposed and obtain different estimates on the linear problems.

In Section 4 we obtain the existence of the inertial manifolds. To accomplish this task we apply the results from [16].

In Section 5 using the implicit definition of the inertial manifolds (given as a fixed point of an appropriate functional) and with the estimates of Section 3 we prove the main result.

2. Setting of the problem and main results

Let A_0 be a self-adjoint positive linear operator on a separable Hilbert space X_0 with domain $D(A_0)$, that is $A_0 : D(A_0) \subset X_0 \to X_0$. We denote by X_0^α, with $\alpha \in [0, 1]$, the fractional power spaces associated to the operator A_0 and $\|\cdot\|_\alpha$ its norm, defined in the usual way, see for instance [11, 8].

We consider the following evolutionary problem,

$$(P_0)\left\{\begin{array}{ll} u_t^0 + A_0 u^0 & = F_0(u^0), \\ u^0(0) \in X_0^\alpha, & \end{array}\right. \tag{2.1}$$

with $F_0 : X_0^\alpha \to X_0$ certain nonlinearity guaranteeing that we have global existence of solutions.

We also consider a perturbed problem,

$$(P_\varepsilon)\left\{\begin{array}{ll} u_t^\varepsilon + A_\varepsilon u^\varepsilon & = F_\varepsilon(u^\varepsilon), \qquad 0 < \varepsilon \le \varepsilon_0 \\ u^\varepsilon(0) \in X_\varepsilon^\alpha, & \end{array}\right. \tag{2.2}$$

where A_ε is also a self-adjoint positive linear operator on a Hilbert space X_ε, that is $A_\varepsilon : D(A_\varepsilon) = X_\varepsilon^1 \subset X_\varepsilon \to X_\varepsilon$, and the nonlinear term $F_\varepsilon : X_\varepsilon^\alpha \to X_\varepsilon$ is another nonlinearity guaranteeing also global existence of solutions of (2.2). We will impose appropriate hypotheses on F_ε and A_ε so such that problem (P_ε) converges to (P_0) as ε tends to 0 in some sense.

Since our aim is to compare different aspects about the dynamic of both problems, (2.1) and (2.2) and these dynamics live in different functional spaces X_0, and X_ε, we will need to compare functions from X_0 and X_ε, (X_0^α and X_ε^α, respectively, with $\alpha \in [0, 1)$ fixed above). So, we assume the existence of linear continuous operators, E and M, such that,

$$E : X_0 \to X_\varepsilon, \qquad \text{and} \quad M : X_\varepsilon \to X_0,$$

and,

$$E_{|X_0^\alpha} : X_0^\alpha \to X_\varepsilon^\alpha, \qquad \text{and} \quad M_{|X_\varepsilon^\alpha} : X_\varepsilon^\alpha \to X_0^\alpha.$$

We will assume they are bounded uniform in ε and without loss of generality we will assume

$$\|E\|_{\mathcal{L}(X_0,X_\varepsilon)}, \|M\|_{\mathcal{L}(X_\varepsilon,X_0)} \le 2, \qquad \|E\|_{\mathcal{L}(X_0^\alpha,X_\varepsilon^\alpha)}, \|M\|_{\mathcal{L}(X_\varepsilon^\alpha,X_0^\alpha)} \le 2. \tag{2.3}$$

We also assume these operators satisfy the following properties,

$$M \circ E = I, \qquad \|Eu_0\|_{X_\varepsilon} \to \|u_0\|_{X_0} \quad \text{for} \quad u_0 \in X_0. \tag{2.4}$$

We will also assume that the family of operators A_ε, for $0 \le \varepsilon \le \varepsilon_0$, have compact resolvent, that is, the resolvent operators are compact for all $\lambda \in \rho(A_\varepsilon)$ where $\rho(A_\varepsilon)$ is the resolvent set of A_ε. This fact, together with the fact that the operators are selfadjoint, implies that its spectrum is discrete real and consists only of eigenvalues, each one with finite multiplicity. Moreover, the fact that A_ε, $0 \le \varepsilon \le \varepsilon_0$, is positive implies that its spectrum is positive. So, we denote by $\sigma(A_\varepsilon)$, the spectrum of the operator A_ε, with,

$$\sigma(A_\varepsilon) = \{\lambda_n^\varepsilon\}_{n=1}^\infty, \qquad \text{and} \quad 0 < c \le \lambda_1^\varepsilon \le \lambda_2^\varepsilon \le \dots \le \lambda_n^\varepsilon \le \dots$$

and we also denote by $\{\varphi_i^\varepsilon\}_{i=1}^\infty$ an associated orthonormal family of eigenfunctions.

With respect to the relation between both operators, A_0 and A_ε we will assume the following hypothesis

(H1). *With α the exponent from problems (2.1) and (2.2), we have*

$$\|A_\varepsilon^{-1} - EA_0^{-1}M\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon^\alpha)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.5}$$

Notice in particular that from (2.5) we also have that $\|A_\varepsilon^{-1}E - EA_0^{-1}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \to 0$ as $\varepsilon \to 0$. Let us define $\tau(\varepsilon)$ as an increasing function of ε such that

$$\|A_\varepsilon^{-1}E - EA_0^{-1}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \le \tau(\varepsilon). \tag{2.6}$$

With respect to the nonlinearities F_0 and F_ε,

(H2). *We assume that the nonlinear terms $F_\varepsilon : X_\varepsilon^\alpha \to X_\varepsilon$ for $0 \le \varepsilon \le \varepsilon_0$, satisfy:*

(a) *They are uniformly bounded, that is, there exists a constant $C_F > 0$ independent of ε such that,*

$$\|F_\varepsilon\|_{L^\infty(X_\varepsilon^\alpha, X_\varepsilon)} \le C_F.$$

(b) *They are globally Lipschitz on X_ε^α with a uniform Lipstichz constant L_F, that is,*

$$\|F_\varepsilon(u) - F_\varepsilon(v)\|_{X_\varepsilon} \le L_F\|u - v\|_{X_\varepsilon^\alpha}. \tag{2.7}$$

(c) *They have a uniform compact support in $\varepsilon \ge 0$: there exists $R > 0$ such that*

$$Supp F_\varepsilon \subset D_R = \{u_\varepsilon \in X_\varepsilon^\alpha : \|u_\varepsilon\|_{X_\varepsilon^\alpha} \le R\}.$$

(d) *F_ε approaches F_0 in the following sense,*

$$\sup_{u_0 \in X_0^\alpha} \|F_\varepsilon(Eu_0) - EF_0(u_0)\|_{X_\varepsilon} = \rho(\varepsilon), \tag{2.8}$$

and $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$.

As we will see below, the convergence of the resolvent operators given by hypothesis **(H1)** guarantees the spectral convergence of the operators, that is, the convergence of the eigenvalues and the eigenfunctions (or eigenprojections). This implies in particular that if we have a gap on the eigenvalues of A_0, we will also have, for ε small enough a similar gap for the eigenvalues of A_ε. This fact, together with the uniform estimates on the nonlinerities F_ε given by hypothesis **(H2)**, guarantees that we may construct inertial manifolds of the same dimension for all $0 \le \varepsilon \le \varepsilon_0$. We will follow the Lyapunov-Perron method, as developed in [16] to obtain these inertial manifolds $\mathcal{M}_\varepsilon$, $0 \le \varepsilon \le \varepsilon_0$. As a matter of fact, consider $m \in \mathbb{N}$ such that $\lambda_m^0 < \lambda_{m+1}^0$ and denote by $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}$ the canonical orthogonal projection onto the eigenfunctions, $\{\varphi_i^\varepsilon\}_{i=1}^m$, corresponding to the first m eigenvalues of the operator A_ε, $0 \le \varepsilon \le \varepsilon_0$ and $\mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}}$ its orthogonal complement, see (3.7) and (3.8). For technical reasons, we express any element belonging to the linear subspace $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(X_\varepsilon)$ in the following basis,

$$\{\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(E\varphi_1^0), \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(E\varphi_2^0), ..., \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(E\varphi_m^0)\}, \qquad \text{for} \quad 0 \le \varepsilon \le \varepsilon_0,$$

with $\{\varphi_i^0\}_{i=1}^m$ the eigenfunctions related to the first m eigenvalues of A_0, which will be seen below that is a basis in $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(X_\varepsilon)$ and in $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(X_\varepsilon^\alpha)$. We will denote by $\psi_i^\varepsilon = \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(E\varphi_i^0)$.

Let us denote by j_ε the isomorphism from $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(X_\varepsilon) = [\psi_1^\varepsilon, ..., \psi_m^\varepsilon]$ onto $\mathbb{R}^m$, that gives us the coordinates of each vector. That is,

$$\begin{aligned} j_\varepsilon : \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(X_\varepsilon) &\longrightarrow \mathbb{R}^m, \\ w_\varepsilon &\longmapsto \bar{p}, \end{aligned} \tag{2.9}$$

where $w_\varepsilon = \sum_{i=1}^m p_i \psi_i^\varepsilon$ and $\bar{p} = (p_1, ..., p_m)$.

We denote by $|\cdot|$ the usual norm in $\mathbb{R}^m$,

$$|\bar{p}| = \left(\sum_{i=1}^{m} p_i^2 \right)^{\frac{1}{2}}, \tag{2.10}$$

and by $|\cdot|_\alpha$ the following one,

$$|\bar{p}|_\alpha = \left(\sum_{i=1}^{m} p_i^2 (\lambda_i^\varepsilon)^{2\alpha} \right)^{\frac{1}{2}}. \tag{2.11}$$

We consider the spaces $(\mathbb{R}^m, |\cdot|)$ and $(\mathbb{R}^m, |\cdot|_\alpha)$, that is, $\mathbb{R}^m$ with the norm $|\cdot|$ and $|\cdot|_\alpha$, respectively, and notice that for $w_0 = \sum_{i=1}^m p_i \psi_i^0$ and $0 \leq \alpha < 1$ we have that,

$$\|w_0\|_{X_0^\alpha} = |j_0(w_0)|_\alpha. \tag{2.12}$$

With this notation, if we define the set $\mathcal{F}_L$ as:

$$\mathcal{F}_L = \{\Phi_\varepsilon : \mathbb{R}^m \to \mathbf{Q}_\mathbf{m}^\varepsilon(X_\varepsilon^\alpha), \quad \text{such that} \quad \|\Phi_\varepsilon(\bar{p}^1) - \Phi_\varepsilon(\bar{p}^2)\|_{X_\varepsilon^\alpha} \leq L|\bar{p}^1 - \bar{p}^2|_\alpha, \quad \bar{p}^1, \bar{p}^2 \in \mathbb{R}^m,$$
$$\text{and} \quad \text{supp } \Phi_\varepsilon \subset B_R\}.$$

Then we can show the following result.

Proposition 2.1. *Let hypotheses* **(H1)** *and* **(H2)** *be satisfied. Assume also that $m \geq 1$ is such that,*

$$\lambda_{m+1}^0 - \lambda_m^0 \geq 12 L_F \left[(\lambda_m^0)^\alpha + (\lambda_{m+1}^0)^\alpha \right], \tag{2.13}$$

and

$$(\lambda_m^0)^{1-\alpha} \geq 24 L_F (1-\alpha)^{-1}. \tag{2.14}$$

Then, there exist $L < 1$ and $\varepsilon_0 > 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_0$ there exists an inertial manifold $\mathcal{M}_\varepsilon$ for (2.1) and (2.2), given by the "graph" of a function $\Phi_\varepsilon \in \mathcal{F}_L$.

Remark 2.1. *i) Observe that the gap condition is stated for the eigenvalues of the limit problem. In particular, this implies that the inertial manifold is obtained of the same dimension m for all values of the parameter $0 \leq \varepsilon \leq \varepsilon_0$.*
ii) We have written quotations in the word "graph" since the manifold $\mathcal{M}_\varepsilon$ is not properly speaking the graph of the function Φ_ε but rather the graph of the appropriate function obtained via the isomorphism j_ε which identifies $\mathbb{R}^m$ with $\mathbf{P}_\mathbf{m}^\varepsilon(X_\varepsilon^\alpha)$. That is,

$$\mathcal{M}_\varepsilon = \{ j_\varepsilon^{-1}(\bar{p}) + \Phi_\varepsilon(\bar{p}); \quad \bar{p} \in \mathbb{R}^m \}$$

The main result we want to show in this article is the following:

Theorem 2.2. *Let hypotheses* **(H1)** *and* **(H2)** *be satisfied and let $\tau(\varepsilon)$ be defined by* (2.6). *Then, under the hypothesis of Proposition 2.1, if Φ_0, Φ_ε are the maps that give us the inertial manifolds, then we have,*

$$\|\Phi_\varepsilon - E\Phi_0\|_{L^\infty(\mathbb{R}^m, X_\varepsilon^\alpha)} \leq C[\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)], \tag{2.15}$$

with C a constant independent of ε.

Remark 2.2. *Observe that the estimate* (2.15) *consists of two terms, $\tau(\varepsilon)|\log(\tau(\varepsilon))|$, inherited from the distance of the resolvent operators and $\rho(\varepsilon)$ inherited from the distance of the nonlinear terms. The factor $|\log(\tau(\varepsilon))|$ seems to appear because of technical reasons. A better estimate would be $\|\Phi_\varepsilon - E\Phi_0\|_{L^\infty(\mathbb{R}^m, X_\varepsilon^\alpha)} \leq C[\tau(\varepsilon) + \rho(\varepsilon)]$, which we have not been able to show, although it is very plausible that this would be true and it should be the optimal rate.*

3. Linear analysis and spectral behavior

The spectral decomposition of the operator A_ε implies that if $\lambda \in \rho(A_\varepsilon)$ then,

$$(\lambda - A_\varepsilon)^{-1}u = \sum_{i=1}^{\infty} \frac{1}{\lambda - \lambda_i^\varepsilon}(u, \varphi_i^\varepsilon)\varphi_i^\varepsilon.$$

In particular, for $\varepsilon \geq 0$,

$$\|(\lambda - A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \leq \max_{i\in\mathbb{N}} \left\{ \frac{1}{|\lambda - \lambda_i^\varepsilon|}, \quad \lambda_i^\varepsilon \in \sigma(A_\varepsilon) \right\} = \frac{1}{dist(\lambda, \sigma(A_\varepsilon))},$$

and if we denote by

$$S_{a,\phi} = \{\lambda \in \mathbb{C} : \phi \leq |arg(\lambda - a)| \leq \pi\},$$

then,

$$\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \leq \frac{C_1}{|\lambda| + 1} \qquad \forall \lambda \in S_{a,\phi},$$

with C_1 independent of ε.

For $\alpha \geq 0$ and for all $0 \leq \varepsilon \leq \varepsilon_0$, let $A_{\varepsilon|_{X_\varepsilon^\alpha}} : X_\varepsilon^{1+\alpha} \subset X_\varepsilon^\alpha \to X_\varepsilon^\alpha$, with domain $X_\varepsilon^{1+\alpha} \subset X_\varepsilon^1$, be the restriction of A_ε to the fractional power space $X_\varepsilon^\alpha \subset X_\varepsilon$ so that,

$$A_\varepsilon u = A_{\varepsilon|_{X_\varepsilon^\alpha}} u \qquad \forall u \in X_\varepsilon^{1+\alpha}.$$

Then $A_{\varepsilon|_{X_\varepsilon^\alpha}}$ is also a sectorial operator on X_ε^α and with a similar spectral decomposition as above, we can also obtain the estimate

$$\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon^\alpha, X_\varepsilon^\alpha)} \leq \frac{1}{dist(\lambda, \sigma(A_\varepsilon))}, \qquad 0 \leq \varepsilon \leq \varepsilon_0.$$

and the following estimates holds, see [8],

$$\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon^\alpha, X_\varepsilon^\alpha)} \leq \frac{C_1}{|\lambda| + 1}, \qquad 0 \leq \varepsilon \leq \varepsilon_0$$

for $\lambda \in S_{a,\phi}$, where $S_{a,\phi}$ is the sector of sectorial property of A_ε and $C_1 > 1$ independent of ε.

Moreover, since A_ε is a sectorial operator, $-A_\varepsilon$ is the infinitesimal generator of a linear semigroup that we denote as $e^{-A_\varepsilon t}$, where,

$$e^{-A_\varepsilon t} = \frac{1}{2\pi i} \int_\Gamma (\lambda I + A_\varepsilon)^{-1} e^{\lambda t} d\lambda,$$

with Γ a contour in the resolvent set of $-A_\varepsilon$, $\rho(-A_\varepsilon)$, with $arg\lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\frac{\pi}{2}, \pi)$, (see [11]). Since A_ε, $\varepsilon \geq 0$, is a self-adjoint operator, the formula above is equivalent to

$$e^{-A_\varepsilon t}u = \sum_{i=1}^{\infty} e^{-\lambda_i^\varepsilon t}(u, \varphi_i^\varepsilon)\varphi_i^\varepsilon. \tag{3.1}$$

Moreover, we have the following result.

Lemma 3.1. *We have the following estimates for the linear semigroup*

$$\|e^{-A_\varepsilon t}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \leq e^{-\lambda_1^\varepsilon t} \leq 1,$$

and,

$$\|e^{-A_\varepsilon t}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon^\alpha)} \leq e^{-\lambda_1^\varepsilon t} \left(\max\{\lambda_1^\varepsilon, \frac{\alpha}{t}\} \right)^\alpha,$$

for $t \geq 0$.

Proof. With the expression of the semigroup given by (3.1), we get

$$\|e^{-A_\varepsilon t}u\|_{X_\varepsilon^\alpha} = \left(\sum_{i=1}^{\infty} e^{-2\lambda_i^\varepsilon t}(u,\varphi_i^\varepsilon)^2(\lambda_i^\varepsilon)^{2\alpha}\right)^{\frac{1}{2}}.$$

The function $f(\lambda) = e^{-\lambda t}\lambda^\alpha$ attains its maximum at $\lambda = \frac{\alpha}{t}$. Then, we have to distinguish two cases:

If $\frac{\alpha}{t} < \lambda_1^\varepsilon$, we obtain

$$\|e^{-A_\varepsilon t}u\|_{X_\varepsilon^\alpha} \leq e^{-\lambda_1^\varepsilon t}(\lambda_1^\varepsilon)^\alpha \|u\|_{X_\varepsilon}.$$

And if $\lambda_1^\varepsilon \leq \frac{\alpha}{t}$,

$$\|e^{-A_\varepsilon t}u\|_{X_\varepsilon^\alpha} \leq e^{-\alpha}\left(\frac{\alpha}{t}\right)^\alpha \|u\|_{X_\varepsilon} \leq e^{-\lambda_1^\varepsilon t}\left(\frac{\alpha}{t}\right)^\alpha \|u\|_{X_\varepsilon}.$$

That is,

$$\|e^{-A_\varepsilon t}u\|_{X_\varepsilon^\alpha} \leq e^{-\lambda_1^\varepsilon t}\left(\max\{\lambda_1^\varepsilon, \frac{\alpha}{t}\}\right)^\alpha \|u\|_{X_\varepsilon}.$$

In the same way, since

$$\|e^{-A_\varepsilon t}u\|_{X_\varepsilon} = \left(\sum_{i=1}^{\infty} e^{-2\lambda_i^\varepsilon t}(u,\varphi_i^\varepsilon)^2\right)^{\frac{1}{2}},$$

then, we obtain,

$$\|e^{-A_\varepsilon t}u\|_{X_\varepsilon} \leq e^{-\lambda_1^\varepsilon t}\|u\|_{X_\varepsilon}.$$

This concludes the proof of the result. ■

With respect to the relation of the spectrum we have the following result.

Lemma 3.2. *If K_0 is a compact set of the complex plane with $K_0 \subset \rho(A_0)$, the resolvent set of A_0, and hypothesis* **(H1)** *is satisfied, then there exists $\varepsilon_0(K_0) > 0$ such that $K_0 \subset \rho(A_\varepsilon)$ for all $0 < \varepsilon \leq \varepsilon_0(K_0)$. Moreover, we have the estimates:*

$$\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon^\alpha)} \leq C(K_0), \qquad \|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \leq C(K_0), \tag{3.2}$$

for all $\lambda \in K_0$, $0 < \varepsilon \leq \varepsilon_0(K_0)$.

Proof. Let us start by showing the following: if $\lambda_{\varepsilon_n} \in \rho(A_{\varepsilon_n})$ with $\|(\lambda_{\varepsilon_n} I - A_{\varepsilon_n})^{-1}\|_{\mathcal{L}(X_{\varepsilon_n}, X_{\varepsilon_n}^\alpha)} \geq k_n$, $k_n \to +\infty$ as $n \to +\infty$, and $\lambda_{\varepsilon_n} \to \lambda_0$, then $\lambda_0 \in \sigma(A_0)$.

Then, assume there exists a sequence $\{\lambda_{\varepsilon_n}\} \in \rho(A_{\varepsilon_n})$ with $\|(\lambda_{\varepsilon_n} I - A_{\varepsilon_n})^{-1}\|_{\mathcal{L}(X_{\varepsilon_n}, X_{\varepsilon_n}^\alpha)} \geq k_n$, and such that $\lambda_{\varepsilon_n} \to \lambda_0$ as $\varepsilon_n \to 0$, for some λ_0. This implies that there exists $f_{\varepsilon_n} \in X_{\varepsilon_n}$ with $\|f_{\varepsilon_n}\|_{X_{\varepsilon_n}} = 1$ and if $w_{\varepsilon_n} = (\lambda_{\varepsilon_n} I - A_{\varepsilon_n})^{-1} f_{\varepsilon_n}$, then $\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha} \to +\infty$.

If we define $u_{\varepsilon_n} = w_{\varepsilon_n}/\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}$, then $\lambda_{\varepsilon_n} u_{\varepsilon_n} - A_{\varepsilon_n} u_{\varepsilon_n} = f_{\varepsilon_n}/\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}$, which implies

$$A_{\varepsilon_n} u_{\varepsilon_n} = \lambda_{\varepsilon_n} u_{\varepsilon_n} - \frac{f_{\varepsilon_n}}{\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}}.$$

Let $\hat{u}_{\varepsilon_n} \in X_0^\alpha$ satisfy the following equation,

$$A_0 \hat{u}_{\varepsilon_n} = \lambda_{\varepsilon_n} M u_{\varepsilon_n} - \frac{M f_{\varepsilon_n}}{\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}}. \tag{3.3}$$

If we study the norm of the right side, since $\left\|\frac{M f_{\varepsilon_n}}{\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}}\right\|_{X_0} \to 0$, we have, by (2.3)

$$\left\|\lambda_{\varepsilon_n} M u_{\varepsilon_n} - \frac{M f_{\varepsilon_n}}{\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}}\right\|_{X_0} \leq 2|\lambda_{\varepsilon_n}|\|u_{\varepsilon_n}\|_{X_{\varepsilon_n}} + \left\|\frac{M f_{\varepsilon_n}}{\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}}\right\|_{X_0} \leq C.$$

So, $\{\hat{u}_{\varepsilon_n}\} \subset X_0^\alpha$ is a compact family. Then, there exists a $\hat{u}_0 \in X_0^\alpha$ and a subsequence, we denote it again as $\hat{u}_{\varepsilon_n}$, such that $\hat{u}_{\varepsilon_n} \to \hat{u}_0$ in X_0^α, as $\varepsilon_n \to 0$. Moreover, by hypothesis (H1), we have, $\|u_{\varepsilon_n} - E\hat{u}_{\varepsilon_n}\|_{X^\alpha_{\varepsilon_n}} \to 0$. And,

$$\|u_{\varepsilon_n} - E\hat{u}_0\|_{X_{\varepsilon_n}} \le \|u_{\varepsilon_n} - E\hat{u}_{\varepsilon_n}\|_{X_{\varepsilon_n}} + \|E\hat{u}_{\varepsilon_n} - E\hat{u}_0\|_{X_{\varepsilon_n}} \le$$

$$\le \|u_{\varepsilon_n} - E\hat{u}_{\varepsilon_n}\|_{X_{\varepsilon_n}} + 2\|\hat{u}_{\varepsilon_n} - \hat{u}_0\|_{X_0} \to 0.$$

So, again by (2.3),

$$\|Mu_{\varepsilon_n} - \hat{u}_0\|_{X_0} = \|M(u_{\varepsilon_n} - E\hat{u}_0)\|_{X_0} \le 2\|u_{\varepsilon_n} - E\hat{u}_0\|_{X_{\varepsilon_n}} \to 0.$$

Hence, via subsequences, $\lambda_{\varepsilon_n} Mu_{\varepsilon_n} - \frac{Mf_{\varepsilon_n}}{\|w_{\varepsilon_n}\|_{X^\alpha_{\varepsilon_n}}} \to \lambda_0 \hat{u}_0$ in X_0 for some $\hat{u}_0 \in X_0^\alpha$. Also, from the definition of u_{ε_n} we have that $\|u_{\varepsilon_n}\|_{X^\alpha_{\varepsilon_n}} = 1$. Then $1 = \|u_{\varepsilon_n}\|_{X^\alpha_{\varepsilon_n}} \le \|u_{\varepsilon_n} - E\hat{u}_0\|_{X^\alpha_{\varepsilon_n}} + \|E\hat{u}_0\|_{X^\alpha_{\varepsilon_n}} \le \|u_{\varepsilon_n} - E\hat{u}_0\|_{X^\alpha_{\varepsilon_n}} + 2\|\hat{u}_0\|_{X_0^\alpha}$. But since $\|u_{\varepsilon_n} - E\hat{u}_0\|_{X^\alpha_{\varepsilon_n}} \to 0$ then $\|\hat{u}_0\|_{X_0^\alpha} > 0$ and hence $\hat{u}_0 \neq 0$. So, from equation (3.3) and the above estimates, we obtain $A_0\hat{u}_0 = \lambda_0\hat{u}_0$, which shows that $\lambda_0 \in \sigma(A_0)$.

Next, we apply this result to prove our lemma. For the first part, we proceed as follows. If $K_0 \cap \sigma(A_\varepsilon)$ is non empty for ε small enough, then there exists a sequence $\varepsilon_n \to 0$ and $\hat{\lambda}_{\varepsilon_n} \in K_0 \cap \sigma(A_{\varepsilon_n})$. Since the spectrum of A_{ε_n} is discrete for all ε_n, for each n we can choose $\lambda_{\varepsilon_n} \in \rho(A_{\varepsilon_n})$ such that $|\lambda_{\varepsilon_n} - \hat{\lambda}_{\varepsilon_n}| < \frac{1}{n}$ and $\|(\lambda_{\varepsilon_n} I - A_{\varepsilon_n})^{-1}\|_{\mathcal{L}(X_{\varepsilon_n}, X^\alpha_{\varepsilon_n})} > k_n$ with $k_n \to +\infty$. Moreover, since K_0 is compact, there is a subsequence $\hat{\lambda}_{\hat{\varepsilon}_n}$ with $\hat{\lambda}_{\hat{\varepsilon}_n} \to \lambda_0$ and $\lambda_0 \in K_0$. Then, we have just proved that, $\lambda_0 \in \sigma(A_0)$. This is a contradiction. So, $K_0 \cap \sigma(A_\varepsilon)$ is empty, and then $K_0 \subset \rho(A_\varepsilon)$ as we wanted to prove.

To obtain the desired estimates, suppose there exist sequences $\{\lambda_n\} \in K_0$ and $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to +\infty$ such that,

$$\|(\lambda_n I - A_{\varepsilon_n})^{-1}\|_{\mathcal{L}(X_{\varepsilon_n}, X^\alpha_{\varepsilon_n})} \ge k_n,$$

with $k_n \to +\infty$. Since K_0 is a compact set, there exists a $\lambda_0 \in K_0$ and a subsequence $\{\lambda_{n_k}\} \in K_0$ with $\lambda_{n_k} \to \lambda_0$, $\lambda_0 \in K_0$, and

$$\|(\lambda_{n_k} I - A_{\varepsilon_{n_k}})^{-1}\|_{\mathcal{L}(X_{\varepsilon_{n_k}}, X^\alpha_{\varepsilon_{n_k}})} \ge k_{n_k}.$$

Then, we have proved above that, $\lambda_0 \in \sigma(A_0)$. This is a contradiction because $\lambda_0 \in K_0 \subset \rho(A_0)$. So, we have for $\lambda \in K_0$,

$$\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon, X^\alpha_\varepsilon)} \le C(K_0), \qquad \|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \le C(K_0).$$

This concludes the proof. ∎

Remark 3.3. *The result just proved implies the uppersemicontinuity of the spectrum: if $\lambda_\varepsilon \in \sigma(A_\varepsilon)$ and $\lambda_\varepsilon \to \lambda_0$ (via subsequences) then $\lambda_0 \in \sigma(A_0)$.*

Now we want to estimate $\|(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\|_{\mathcal{L}(X_0, X^\alpha_\varepsilon)}$. We have the following result.

Lemma 3.4. *With the notation above, if $\lambda \in \rho(-A_0)$ and ε is small enough so that $\lambda \in \rho(-A_\varepsilon)$ and hypothesis* **(H1)** *is satisfied then,*

$$\|(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\|_{\mathcal{L}(X_0, X^\alpha_\varepsilon)} \le C_3^\varepsilon(\lambda)\tau(\varepsilon),$$

where $C_3^\varepsilon(\lambda) = \left(1 + \frac{|\lambda|}{dist(\lambda, \sigma(-A_\varepsilon))}\right)\left(1 + \frac{|\lambda|}{dist(\lambda, \sigma(-A_0))}\right)$ and $\tau(\varepsilon)$ is defined by (2.6).

Proof. First of all notice that from Lemma 3.2 if $\lambda \in \rho(-A_0)$ then $\lambda \in \rho(-A_\varepsilon)$ for ε small enough. Hence $(\lambda I + A_\varepsilon)^{-1}$ and $(\lambda I + A_0 I)^{-1}$ are well defined for all $\lambda \in \rho(-A_0)$.

We are interested in estimating,

$$\|(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)}.$$

The first thing we are going to do is to show the following identity:

$$(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1} = [I - (\lambda I + A_\varepsilon)^{-1}\lambda](A_\varepsilon^{-1}E - EA_0^{-1})[I - \lambda(\lambda I + A_0)^{-1}]. \tag{3.4}$$

First, note that

$$(I + A_\varepsilon^{-1}\lambda)[I - (A_\varepsilon + \lambda I)^{-1}\lambda] = I, \tag{3.5}$$

then,

$$(I + A_\varepsilon^{-1}\lambda)(\lambda I + A_\varepsilon)^{-1} = A_\varepsilon^{-1}.$$

Hence,

$$\begin{aligned}(I + A_\varepsilon^{-1}\lambda)\left[(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\right] = \\ = A_\varepsilon^{-1}E - E(\lambda I + A_0)^{-1} - A_\varepsilon^{-1}\lambda E(\lambda I + A_0)^{-1}.\end{aligned}$$

Since,

$$\begin{aligned}E(\lambda I + A_0)^{-1} = EA_0^{-1} - EA_0^{-1} + E(\lambda I + A_0)^{-1} = EA_0^{-1} - EA_0^{-1}[I - A_0(\lambda I + A_0)^{-1}] = \\ = EA_0^{-1} - EA_0^{-1}[(A_0 + \lambda I)^{-1}\lambda],\end{aligned}$$

we have,

$$\begin{aligned}(I + A_\varepsilon^{-1}\lambda)\left[(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\right] = \\ = A_\varepsilon^{-1}E - A_\varepsilon^{-1}E\lambda(\lambda I + A_0)^{-1} - EA_0^{-1} + EA_0^{-1}[(A_0 + \lambda I)^{-1}\lambda] = \\ = (A_\varepsilon^{-1}E - EA_0^{-1})[I - \lambda(\lambda I + A_0)^{-1}].\end{aligned}$$

By (3.5), $[I - \lambda(A_\varepsilon + \lambda I)^{-1}](I + A_\varepsilon^{-1}\lambda) = I$, then we obtain the desired identity (3.4),

$$(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1} = [I - \lambda(A_\varepsilon + \lambda I)^{-1}](A_\varepsilon^{-1}E - EA_0^{-1})[I - \lambda(\lambda I + A_0)^{-1}].$$

Hence, since hypothesis (H1) is satisfied, we obtain the desired estimates,

$$\begin{aligned}\|(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq \\ \leq \|(I - \lambda(A_\varepsilon + \lambda I)^{-1}\|_{\mathcal{L}(X_\varepsilon^\alpha, X_\varepsilon^\alpha)}\|A_\varepsilon^{-1}E - EA_0^{-1}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)}\|I - \lambda(\lambda I + A_0)^{-1}\|_{\mathcal{L}(X_0, X_0)} \leq \\ \leq \left(1 + \frac{|\lambda|}{dist(\lambda, \sigma(-A_\varepsilon))}\right)\tau(\varepsilon)\left(1 + \frac{|\lambda|}{dist(\lambda, \sigma(-A_0))}\right).\end{aligned}$$

This concludes the proof. ■

We can easily show now,

Corolary 3.5. *(i) If $K_0 \subset \rho(-A_0)$ as in Lemma 3.2 and $\Sigma_{-a,\phi}$ is the set of the complex plane described by*

$$\Sigma_{-a,\phi} = \{\lambda \in \mathbb{C} : |arg(\lambda + a)| \leq \pi - \phi\},$$

then,

$$\sup_{\lambda \in K_0 \cup \Sigma_{-a,\phi}} C_3^\varepsilon(\lambda) \leq \bar{C}_3.$$

(ii) If we take $a = 0$ and $\phi = \frac{\pi}{4}$ then

$$C_3^\varepsilon(\lambda) \leq \left(1 + \frac{1}{\sin(\phi)}\right)^2 \leq 6, \qquad \text{for all} \quad \lambda \in \Sigma_{0,\frac{\pi}{4}}. \tag{3.6}$$

Remark 3.6. *Note that, although $C_3^\varepsilon(\lambda)$ depends on ε, thanks to the continuity of the eigenvalues, see Remark 3.3, we can consider it uniform in ε.*

The estimate found in Lemma 3.4 will be applied to obtain estimates on the distance of the spectral projections and estimates on the distance of the linear semigroups generated by A_0 and A_ε. Let us start with the spectral projections.

Let us assume that for some $m = 1, 2, ...$ we have $\lambda_m^0 < \lambda_{m+1}^0$ and as we have mentioned in the introduction, we denote by $\{\varphi_i^\varepsilon\}_{i=1}^m$ the first m eigenfunctions of the operator A_ε, $0 \leq \varepsilon \leq \varepsilon_0$ and by $\mathbf{P_m^\varepsilon}$ the canonical orthogonal projection onto the subspace $[\varphi_1^\varepsilon, \dots, \varphi_m^\varepsilon]$, that is, if $0 < \varepsilon \leq \varepsilon_0$

$$\begin{aligned} \mathbf{P_m^\varepsilon} : X_\varepsilon &\longrightarrow X_\varepsilon \\ v &\longrightarrow \mathbf{P_m^\varepsilon}(v) = \textstyle\sum_{i=1}^m (v, \varphi_i^\varepsilon)_{X_\varepsilon} \varphi_i^\varepsilon \end{aligned} \tag{3.7}$$

or if $\varepsilon = 0$,

$$\begin{aligned} \mathbf{P_m^0} : X_0 &\longrightarrow X_0 \\ v &\longrightarrow \mathbf{P_m^0}(v) = \textstyle\sum_{i=1}^m (v, \varphi_i^0)_{X_0} \varphi_i^0 \end{aligned} \tag{3.8}$$

Notice that in a natural way, the projections may be defined in the intermediate space X_ε^α and, since it is a finite linear combination of eigenfunctions, its range is contained also in X_ε^α.

We have the following estimate.

Lemma 3.7. *Let $\{\mathbf{P_m^\varepsilon}\}_{0 \leq \varepsilon \leq \varepsilon_0}$ be the family of canonical orthogonal projections described above, $v \in X_0$, Γ a curve in the complex plane which contains the first m eigenvalues of $-A_0$ and hypothesis (H1) be satisfied. Then,*

$$\|\mathbf{P_m^\varepsilon} E(v) - E\mathbf{P_m^0}(v)\|_{X_\varepsilon^\alpha} \leq C_P \tau(\varepsilon) \|v\|_{X_0},$$

with $C_P = \frac{|\Gamma|}{2\pi} \sup_{\lambda \in \Gamma} C_3^\varepsilon(\lambda)$, $|\Gamma|$ the length of the curve Γ and C_3^ε is given in Lemma 3.4.

Proof. Let Γ be the curve mentioned above. From Lemma 3.2, taking $K_0 = \Gamma$, we have that $\Gamma \subset \rho(-A_\varepsilon)$ for $0 \leq \varepsilon \leq \varepsilon_0(\Gamma)$ with $\varepsilon_0(\Gamma)$ small enough. The spectral projection over the eigenspace generated by the part of the spectrum of $-A_\varepsilon$ contained "inside" the curve Γ is given by

$$\mathbf{P_\Gamma^\varepsilon} = \frac{1}{2\pi i} \int_\Gamma (A_\varepsilon + \lambda I)^{-1} d\lambda, \qquad \text{with} \quad \lambda \in \Gamma, \quad 0 \leq \varepsilon \leq \varepsilon_0.$$

Therefore,

$$\|\mathbf{P_\Gamma^\varepsilon} E(v) - E\mathbf{P_\Gamma^0}(v)\|_{X_\varepsilon^\alpha} \leq \left| \frac{1}{2\pi i} \right| \left| \int_\Gamma \|(\lambda I + A_\varepsilon)^{-1} E(v) - E(\lambda I + A_0)^{-1}(v)\|_{X_\varepsilon^\alpha} d\lambda \right|.$$

Applying now Lemma 3.4, we obtain

$$\|\mathbf{P_\Gamma^\varepsilon} E(v) - E\mathbf{P_\Gamma^0}(v)\|_{X_\varepsilon^\alpha} \leq \frac{1}{2\pi} |\Gamma| \sup_{\lambda \in \Gamma} C_3(\lambda) \tau(\varepsilon) \|v\|_{X_0} = C_P \tau(\varepsilon) \|v\|_{X_0}. \tag{3.9}$$

Since the curve Γ encircles only the first m eigenvalues of $-A_0$, then we know that $\mathbf{P_\Gamma^0} = \mathbf{P_m^0}$, that is, the projection over the first m eigenfunctions. This implies that $Rank(\mathbf{P_\Gamma^0}) = m$ and from (3.9), we also have that $Rank(\mathbf{P_\Gamma^\varepsilon}) = m$ and therefore we also have $\mathbf{P_\Gamma^\varepsilon} = \mathbf{P_m^\varepsilon}$. Hence, (3.9) proves the result. ■

Remark 3.8. *If we have the gap $\lambda_{m+1}^\varepsilon - \lambda_m^\varepsilon \geq 2$, which is not very restrictive in light of conditions* (2.13)*, we construct the curve Γ as the rectangle which contains the first m eigenvalues of $-A_0$ described as follows,*

$$\Gamma = \Gamma^1 \cup \Gamma^2 \cup \Gamma^3 \cup \Gamma^4,$$

where,

$$\begin{gathered} \Gamma^1 = \{\lambda \in \mathbb{C} : Re(\lambda) = -\lambda_1^0 + 1 \text{ and } |Im(\lambda)| \leq 1\}, \\ \Gamma^2 = \{\lambda \in \mathbb{C} : -\lambda_m^0 - 1 \leq Re(\lambda) \leq -\lambda_1^0 + 1 \text{ and } Im(\lambda) = 1\}, \\ \Gamma^3 = \{\lambda \in \mathbb{C} : Re(\lambda) = -\lambda_m^0 - 1 \text{ and } |Im(\lambda)| \leq 1\}, \end{gathered}$$

and,

$$\Gamma^4 = \{\lambda \in \mathbb{C} : -\lambda_m^0 - 1 \leq Re(\lambda) \leq -\lambda_1^0 + 1 \text{ and } Im(\lambda) = -1\}.$$

By Lemma 3.2, taking $\varepsilon \geq 0$ small enough, the rectangle Γ contains the first m eigenvalues $\{\lambda_i^\varepsilon\}_{i=1}^m$. Then, it is easy to see that

$$C_P \leq 24(\lambda_m^0)^3.$$

We can also obtain good estimates for the linear semigroups.

Lemma 3.9. *Let hypothesis* **(H1)** *be satisfied. If we denote,*

$$l_\varepsilon^\alpha(t) := \min\{t^{-1}\tau(\varepsilon), t^{-\alpha}\}, \qquad t > 0 \quad \textit{and} \quad \alpha \in [0,1)$$

then,

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq 3l_\varepsilon^\alpha(t). \tag{3.10}$$

Proof. Let $\Sigma_{0,\phi} = \{\lambda \in \mathbb{C} : |arg(\lambda)| \leq \pi - \phi\}$, with $\phi = \frac{\pi}{4}$, and let Γ be the boundary of $\Sigma_{0,\frac{\pi}{4}}$, that is the curve consisting of the following segments Γ^1 and Γ^2,

$$\Gamma = \Gamma^1 \cup \Gamma^2 = \{re^{-i(\pi-\phi)} : -\infty < r \leq 0\} \cup \{re^{i(\pi-\phi)} : 0 \leq r < +\infty\}$$

oriented such that the imaginary part grows as λ runs in Γ. We know that,

$$e^{-A_\varepsilon t}E - Ee^{-A_0 t} = \frac{1}{2\pi i}\int_\Gamma \left((\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\right)e^{\lambda t}d\lambda.$$

So,

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq \frac{1}{2\pi}\left|\int_\Gamma C_3\tau(\varepsilon)|e^{\lambda t}|d\lambda\right|,$$

with $C_3 = \sup_{\lambda\in\Gamma} C_3(\lambda)$. Since $\lambda \in \Gamma$,

$$|e^{\lambda t}| = |e^{(-re^{-i(\pi-\phi)})t}| = e^{(rcos(\phi))t} \qquad \text{for} \quad -\infty \leq r \leq 0, \quad \lambda \in \Gamma^1$$

and,

$$|e^{\lambda t}| = |e^{(re^{i(\pi-\phi)})t}| = e^{(-rcos(\phi))t} \qquad \text{for} \quad 0 \leq r \leq +\infty, \quad \lambda \in \Gamma^2.$$

With this,

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq \frac{2}{2\pi}C_3\tau(\varepsilon)\int_0^\infty e^{(-rcos(\phi))t}dr.$$

We make the change of variables $(rcos(\phi))t = z$, and then,

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq \frac{1}{\pi}C_3\tau(\varepsilon)\frac{1}{cos(\phi)t}\int_0^\infty e^{-z}dz \leq \frac{1}{\pi cos(\phi)}C_3\tau(\varepsilon)t^{-1},$$

with $C_3 = \sup_{\lambda\in\Gamma} C_3(\lambda) \leq 6$ and, for $\phi = \frac{\pi}{4}$, $\frac{C_3}{\pi cos(\phi)} < 3$.

On the other hand,

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq \|e^{-A_\varepsilon t}E\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} + \|Ee^{-A_0 t}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha))}.$$

Then, by Lemma 3.1 and (2.3),

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq 2e^{-\lambda_1^\varepsilon t}\left(\max\{\lambda_1^\varepsilon, \frac{\alpha}{t}\}\right)^\alpha + 2e^{-\lambda_1^\varepsilon t}\left(\max\{\lambda_1^\varepsilon, \frac{\alpha}{t}\}\right)^\alpha \leq 4e^{-\lambda_1^\varepsilon t}\left(\max\{\lambda_1^\varepsilon, \frac{\alpha}{t}\}\right)^\alpha.$$

This shows the result. ∎

For further analysis we will include here some properties of the function $l_\varepsilon^\alpha(t)$ that will be used below.

Lemma 3.10. *Let $0 \le \gamma < 1$ and $a > 0$. If we consider, for all $t > 0$,*

$$l_\varepsilon^\alpha(t) := \min\{t^{-1}\tau(\varepsilon), t^{-\alpha}\}, \qquad \textit{with} \quad 0 \le \alpha < 1, \quad \textit{and} \quad \tau(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0,$$

then, we have the following estimates,

$$\int_0^t (t-s)^{-\gamma} l_\varepsilon^\alpha(s)ds \le \frac{2^\gamma}{(1-\gamma)(1-\alpha)} t^{-\gamma}(|\log(t)) + |\log(\tau(\varepsilon))|)\tau(\varepsilon),$$

$$\int_0^t e^{-as} l_\varepsilon^\alpha(s)ds \le \frac{2}{1-\alpha}(|\log(t)| + |\log(\tau(\varepsilon))|)\tau(\varepsilon),$$

and,

$$\int_0^\infty e^{-as} l_\varepsilon^\alpha(s)ds \le \frac{2}{1-\alpha}|\log(\tau(\varepsilon))|\tau(\varepsilon), \qquad \textit{if} \quad a \ge 1.$$

Proof. To prove the first estimate, we divide the analysis in several cases. First, if $0 < t \le 2h(\varepsilon)^{\frac{1}{1-\alpha}}$, we have

$$\int_0^t (t-s)^{-\gamma} l_\varepsilon^\alpha(s)ds \le \int_0^t (t-s)^{-\gamma} s^{-\alpha} ds = t^{-\gamma+1-\alpha} \int_0^1 (1-z)^{-\gamma} z^{-\alpha} dz$$

where we have performed the change of variables $s = tz$ in the integral. Hence,

$$\int_0^t (t-s)^{-\gamma} l_\varepsilon^\alpha(s)ds \le Ct^{-\gamma} t^{1-\alpha} \le Ct^{-\gamma} h(\varepsilon).$$

Second, if $2h(\varepsilon)^{\frac{1}{1-\alpha}} \le t$, then

$$\int_0^t (t-s)^{-\gamma} l_\varepsilon^\alpha(s)ds \le \int_0^{h(\varepsilon)^{\frac{1}{1-\alpha}}} (t-s)^{-\gamma} s^{-\alpha} ds + \int_{h(\varepsilon)^{\frac{1}{1-\alpha}}}^{t/2} (t-s)^{-\gamma} s^{-1} h(\varepsilon) ds + \int_{t/2}^t (t-s)^{-\gamma} s^{-1} h(\varepsilon) ds =$$
$$I_1 + I_2 + I_3.$$

We study each term separately. For the first one, I_1, note that if $t \ge 2h(\varepsilon)^{\frac{1}{1-\alpha}}$ and $s \in [0, h(\varepsilon)^{\frac{1}{1-\alpha}}]$ then $t - s \ge \frac{t}{2}$. So,

$$I_1 \le \left(\frac{t}{2}\right)^{-\gamma} \int_0^{h(\varepsilon)^{\frac{1}{1-\alpha}}} s^{-\alpha} ds \le 2^\gamma t^{-\gamma} \frac{1}{1-\alpha} h(\varepsilon),$$

$$I_2 \le (t/2)^{-\gamma}(\log(t/2) - \log(h(\varepsilon)^{\frac{1}{1-\alpha}}))h(\varepsilon) \le 2^\gamma t^{-\gamma}(|\log(t)| + \frac{1}{1-\alpha}|\log(h(\varepsilon))|)h(\varepsilon),$$

$$I_3 \le t^{-\gamma} \int_{1/2}^1 (1-z)^{-\gamma} z^{-1} dz h(\varepsilon) \le \frac{2^\gamma}{1-\gamma} t^{-\gamma} h(\varepsilon) \le \frac{2^\gamma}{1-\gamma}\frac{1}{1-\alpha} t^{-\gamma} h(\varepsilon).$$

Putting together the three estimates we show the desired estimate,

$$\int_0^t (t-s)^{-\gamma} l_\varepsilon^\alpha(s)ds \le \frac{2^\gamma}{(1-\gamma)(1-\alpha)} t^{-\gamma}(|\log(t)| + |\log(h(\varepsilon))|)h(\varepsilon).$$

For the second estimate, we proceed as follows,

$$\int_0^t e^{-as} l_\varepsilon^\alpha(s)ds = \int_0^{h(\varepsilon)^{\frac{1}{1-\alpha}}} e^{-as} s^{-\alpha} ds + \int_{h(\varepsilon)^{\frac{1}{1-\alpha}}}^t e^{-as} s^{-1} h(\varepsilon) \le$$
$$\le \frac{1}{1-\alpha} h(\varepsilon) + e^{-ah(\varepsilon)^{\frac{1}{1-\alpha}}} h(\varepsilon) \left|\log(t) - \left(\frac{1}{1-\alpha}\right)\log(h(\varepsilon))\right| \le$$
$$\le \frac{2}{1-\alpha}(|\log(t)| + |\log(h(\varepsilon))|)h(\varepsilon),$$

as we wanted to prove. For the last one, we write,

$$\int_0^\infty e^{-as} l_\varepsilon^\alpha(s)ds = \int_0^{h(\varepsilon)^{\frac{1}{1-\alpha}}} s^{-\alpha} ds + \int_{h(\varepsilon)^{\frac{1}{1-\alpha}}}^1 s^{-1} h(\varepsilon) ds + h(\varepsilon) \int_1^\infty e^{-as} s^{-1} ds =$$

$$= \frac{h(\varepsilon)}{1-\alpha} + \frac{1}{1-\alpha}|\log(h(\varepsilon))|h(\varepsilon) + \frac{e^{-a}}{a}h(\varepsilon) \leq \frac{2e^{-a}}{a(1-\alpha)}|\log(h(\varepsilon))|h(\varepsilon).$$

Note that, if $a \geq 1$ then,

$$\int_0^\infty e^{-as} l_\varepsilon^\alpha(s)ds \leq \frac{2}{1-\alpha}|\log(h(\varepsilon))|h(\varepsilon).$$

This concludes the proof of the result. ■

Remark 3.11. *If $t = 1$, the first estimate is simplified to*

$$\int_0^t (t-s)^{-\gamma} l_\varepsilon^\alpha(s)ds \leq \frac{2^\gamma}{(1-\gamma)(1-\alpha)}|\log(h(\varepsilon))|h(\varepsilon). \tag{3.11}$$

4. Existence of Inertial Manifolds

Our objective in this section is to construct inertial manifolds $\mathcal{M}_\varepsilon$, for each $0 \leq \varepsilon \leq \varepsilon_0$, which will be invariant manifolds for the semi flow generated by (2.1) and (2.2), therefore proving Proposition 2.1. For this purpose, we will use the Lyapunov-Perron method, see [16]. This method consists in constructing the inertial manifold as the graph of a Lipschitz map, which is obtained as the fixed point of an appropriate transformation. For that, observe that Lemma 3.2 and Remark 3.3 give us that if the operator A_0 has spectral gap, then the operator A_ε will also have it for ε small enough. This spectral gap is essential in the construction of the inertial manifold.

To obtain these inertial manifolds $\mathcal{M}_\varepsilon$, $0 \leq \varepsilon \leq \varepsilon_0$, consider $m \in \mathbb{N}$ such that $\lambda_m^0 < \lambda_{m+1}^0$ (and therefore $\lambda_m^\varepsilon < \lambda_{m+1}^\varepsilon$ for ε small enough) and denote by $\mathbf{P_m^\varepsilon}$ the canonical orthogonal projection onto the eigenfunctions, $\{\varphi_i^\varepsilon\}_{i=1}^m$, corresponding to the first m eigenvalues of the operator A_ε, $0 \leq \varepsilon \leq \varepsilon_0$ and $\mathbf{Q_m^\varepsilon}$ its orthogonal complement, see (3.7) and (3.8). The Lyapunov-Perron method obtains $\mathcal{M}_\varepsilon$ as the graph of a function $\Psi_\varepsilon : \mathbf{P_m^\varepsilon} X_\varepsilon^\alpha \to \mathbf{Q_m^\varepsilon} X_\varepsilon^\alpha$ which is obtained as a fixed point of the functional

$$(\mathbf{T_\varepsilon}\Psi_\varepsilon)(p^0) = \int_{-\infty}^0 e^{A_\varepsilon \mathbf{Q_m^\varepsilon} s}\mathbf{Q_m^\varepsilon} F_\varepsilon(p(s) + \Psi_\varepsilon(p(s)))ds, \tag{4.1}$$

where $p(s) \in [\varphi_1^\varepsilon, \ldots, \varphi_m^\varepsilon]$ is the globally defined solution of

$$\begin{cases} p_t = -A_\varepsilon p + \mathbf{P_m^\varepsilon} F_\varepsilon(p + \Psi_\epsilon(p(t))) \\ p(0) = p^0. \end{cases} \tag{4.2}$$

Following [16] it can be seen that:

Proposition 4.1. *Assume hypotheses* **(H1)** *and* **(H2)** *are satisfied. If m is such that*

$$\lambda_{m+1}^0 - \lambda_m^0 \geq 12L_F[(\lambda_{m+1}^0)^\alpha + (\lambda_m^0)^\alpha]$$

$$(\lambda_m^0)^{1-\alpha} \geq 24L_F(1-\alpha)^{-1}$$

then equation (2.2) has an inertial manifold $\mathcal{M}_\varepsilon$ given as the graph of a Lipschitz function $\Psi_\varepsilon : [\varphi_1^\varepsilon, \ldots, \varphi_m^\varepsilon] \to \mathbf{Q_m^\varepsilon} X_\varepsilon$ satisfying

$$supp(\Psi_\varepsilon) \subset \{\phi \in \mathbf{P_m^\varepsilon} X_\varepsilon^\alpha, \|\phi\|_{X_\varepsilon^\alpha} \leq R\}$$

$$\|\Psi_\varepsilon(p)\|_{X_\varepsilon^\alpha} \leq L_0$$

$$\|\Psi_\varepsilon(p) - \Psi_\varepsilon(p')\|_{X_\varepsilon^\alpha} \leq L_1\|p - p'\|_{X_\varepsilon^\alpha}$$

for certain L_0, L_1 independent of ε.

Proof. Observe that if m is such that the gap conditions of the proposition hold, then for ε small enough we have

$$\begin{aligned} &(\lambda_m^\varepsilon)^{1-\alpha} \geq 12L_F(1-\alpha)^{-1} \\ &\lambda_{m+1}^\varepsilon - \lambda_m^\varepsilon \geq 6L_F[(\lambda_{m+1}^\varepsilon)^\alpha + (\lambda_m^\varepsilon)^\alpha] \end{aligned} \tag{4.3}$$

which are the gap conditions needed in [16] to obtain the inertial manifolds for each ε small enough. ■

With the definition of the isomorphism j_ε, (2.9), we may define now the inertial manifolds $\Phi_\varepsilon : \mathbb{R}^m \to \mathbf{Q}_\mathbf{m}^\varepsilon X_\varepsilon^\alpha$ as $\Phi_\varepsilon = \Psi_\varepsilon \circ j_\varepsilon^{-1}$. Notice also that since Ψ_ε is a fixed point of $\mathbf{T}_{\boldsymbol{\varepsilon}}$, then the function Φ_ε satisfies,

$$(\mathbf{T}_{\boldsymbol{\varepsilon}}\Phi_\varepsilon)(\bar{p}^0) = \int_{-\infty}^0 e^{A_\varepsilon \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} s}\mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} F_\varepsilon\big(p(s) + \Phi_\varepsilon(j_\varepsilon(p(s)))\big)ds, \tag{4.4}$$

where $p(s)$ is the solution of (4.2) with $p^0 = j_\varepsilon^{-1}(\bar{p}^0)$ or equivalently, $p(s)$ is the solution of

$$\begin{cases} p_t = -A_\varepsilon p + \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} F_\varepsilon(p + \Phi_\epsilon \circ j_\varepsilon(p(t))) \\ p(0) = j_\varepsilon^{-1}(\bar{p}^0). \end{cases} \tag{4.5}$$

It is an easy exercise now to show that these functions Φ_ε are the inertial manifolds from Proposition 2.1.

5. Rate of convergence of the inertial manifolds

Once we have proved the existence of the inertial manifolds $\mathcal{M}_\varepsilon$, $\varepsilon \geq 0$ and therefore we have fixed the value of m, we are interested in obtaining the rate of convergence of these inertial manifolds as $\varepsilon \to 0$. To accomplish this, we will need to subtract the integral expressions (4.4) for $\varepsilon = 0$ and $\varepsilon > 0$ and make several estimates on these differences. Therefore, we will need first to obtain good estimates on the behavior of the semigroup acting in the spaces $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha$ and $\mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha$.

Lemma 5.1. *Let hypothesis* **(H1)** *be satisfied and* Γ *a curve in the complex plane which contains the first* m *eigenvalues of* $-A_0$. *Then,*

$$\|e^{-A_\varepsilon t}\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} E - Ee^{-A_0 t}\mathbf{P}_\mathbf{m}^\mathbf{0}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq C_4 e^{-(\lambda_m^0+\nu)t}\tau(\varepsilon), \qquad t \leq 0,$$

with $C_4 = \frac{|\Gamma|}{2\pi}\sup_{\lambda\in\Gamma} C_3^\varepsilon(\lambda)$.

Proof. Let us consider the curve Γ as the rectangle which contains the first m eigenvalues of $-A_0$ described as follows,

$$\Gamma = \Gamma^1 \cup \Gamma^2 \cup \Gamma^3 \cup \Gamma^4,$$

where,

$$\begin{gathered} \Gamma^1 = \{\lambda \in \mathbb{C} : Re(\lambda) = -\lambda_1^0 + \nu \text{ and } |Im(\lambda)| \leq 1\}, \\ \Gamma^2 = \{\lambda \in \mathbb{C} : -\lambda_m^0 - \nu \leq Re(\lambda) \leq -\lambda_1^0 + \nu \text{ and } Im(\lambda) = 1\}, \\ \Gamma^3 = \{\lambda \in \mathbb{C} : Re(\lambda) = -\lambda_m^0 - \nu \text{ and } |Im(\lambda)| \leq 1\}, \end{gathered}$$

and,

$$\Gamma^4 = \{\lambda \in \mathbb{C} : -\lambda_m^0 - \nu \leq Re(\lambda) \leq -\lambda_1^0 + \nu \text{ and } Im(\lambda) = -1\}.$$

We know that,

$$e^{-A_\varepsilon t}\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} E - Ee^{-A_0 t}\mathbf{P}_\mathbf{m}^\mathbf{0} = \frac{1}{2\pi i}\int_\Gamma \left((\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\right)e^{\lambda t}d\lambda.$$

So,

$$\|e^{-A_\varepsilon t}\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} E - Ee^{-A_0 t}\mathbf{P}_\mathbf{m}^\mathbf{0}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq \frac{1}{2\pi}\int_\Gamma \|(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)}|e^{\lambda t}|d\lambda.$$

Applying Lemma 3.4, for $t \leq 0$ we have,

$$\|e^{-A_\varepsilon t}\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} E - Ee^{-A_0 t}\mathbf{P}_\mathbf{m}^\mathbf{0}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq \frac{|\Gamma|}{2\pi}\sup_{\lambda\in\Gamma} C_3^\varepsilon(\lambda)\tau(\varepsilon)\sup_{\lambda\in\Gamma} e^{Re(\lambda)t} = C_4 e^{-(\lambda_m^0+\nu)t}\tau(\varepsilon),$$

with $C_4 = \frac{|\Gamma|}{2\pi}\sup_{\lambda\in\Gamma} C_3^\varepsilon(\lambda)$ and $|\Gamma|$ the length of the curve Γ.

■

With respect to the behavior of the linear semigroup in the subspace $\mathbf{Q^\varepsilon_m} X^\alpha_\varepsilon$, notice that we have the expression

$$e^{-A_\varepsilon \mathbf{Q^\varepsilon_m} t} u = \sum_{i=m+1}^{\infty} e^{-\lambda_i^\varepsilon t}(u, \varphi_i^\varepsilon)\varphi_i^\varepsilon.$$

Hence, following a similar proof as Lemma 3.1, we get

$$\|e^{-A_\varepsilon \mathbf{Q^\varepsilon_m} t}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \leq e^{-\lambda^\varepsilon_{m+1} t},$$

and,

$$\|e^{-A_\varepsilon \mathbf{Q^\varepsilon_m} t}\|_{\mathcal{L}(X_\varepsilon, X^\alpha_\varepsilon)} \leq e^{-\lambda^\varepsilon_{m+1} t} \left(\max\{\lambda^\varepsilon_{m+1}, \frac{\alpha}{t}\}\right)^\alpha, \tag{5.1}$$

for $t \geq 0$.

Before continuing, we now present technical lemmas henceforward needed.

Lemma 5.2. *Let a be a positive constant, $a > 0$, $\alpha \in (0,1)$ and $\lambda > 0$ a positive real number. We have the following estimate,*

$$\int_0^\infty e^{-as} \left(\max\{\lambda, \frac{\alpha}{s}\}\right)^\alpha ds \leq (1-\alpha)^{-1}\lambda^{\alpha-1} + \lambda^\alpha a^{-1}.$$

Proof. Let $\alpha \in (0,1)$ and λ a real positive number. Then we know that

$$\max\{\lambda, \frac{\alpha}{s}\} = \begin{cases} \frac{\alpha}{s} & \text{if} \quad 0 < s \leq \frac{\alpha}{\lambda} \\ \lambda & \text{if} \quad \frac{\alpha}{\lambda} < s < \infty. \end{cases}$$

So,

$$\int_0^\infty \left(\max\{\lambda, \frac{\alpha}{s}\}\right)^\alpha e^{-as} ds = \int_0^{\frac{\alpha}{\lambda}} \left(\frac{\alpha}{s}\right)^\alpha e^{-as} ds + \int_{\frac{\alpha}{\lambda}}^\infty \lambda^\alpha e^{-as} ds =$$

$$= \alpha^\alpha \int_0^{\frac{\alpha}{\lambda}} s^{-\alpha} e^{-as} ds + \lambda^\alpha \int_{\frac{\alpha}{\lambda}}^\infty e^{-as} ds =$$

$$= \alpha^\alpha \left(\frac{\alpha}{\lambda}\right)^{1-\alpha} (1-\alpha)^{-1} + \lambda^\alpha e^{-\frac{a\alpha}{\lambda}} a^{-1} \leq$$

$$\leq (1-\alpha)^{-1}\lambda^{\alpha-1} + \lambda^\alpha a^{-1},$$

as we wanted to prove. ■

Now, with respect to the comparison of both semigroups $e^{-A_\varepsilon t}$ and $e^{-A_0 t}$ in $\mathbf{Q^\varepsilon_m} X^\alpha_\varepsilon$ and $\mathbf{Q^0_m} X^\alpha_0$, we have the following estimates,

Lemma 5.3. *Let hypothesis* **(H1)** *be satisfied. If, for $t > 0$, as before we denote by*

$$l^\alpha_\varepsilon(t) := \min\{t^{-1}\tau(\varepsilon), t^{-\alpha}\},$$

then, for each $\nu > 0$ small, $m \in \mathbb{N}$ and $t > 0$,

$$\|e^{-A_\varepsilon t}\mathbf{Q^\varepsilon_m} E - E e^{-A_0 t}\mathbf{Q^0_m}\|_{\mathcal{L}(X_0, X^\alpha_\varepsilon)} \leq 3e^{-(\lambda^0_{m+1}-\nu)t} l^\alpha_\varepsilon(t).$$

Proof. From Lemma 3.2 and Remark 3.3, we know that there is a real number $\varepsilon_0 = \varepsilon_0(m)$ such that, for $0 \leq \varepsilon \leq \varepsilon_0$, there is a gap between the mth-eigenvalue, $-\lambda_m^\varepsilon$, and $m+1$-eigenvalues, $-\lambda_{m+1}^\varepsilon$, of $-A_\varepsilon$. We denote by Γ_m the boundary of $\Sigma_{b,\phi} = \{\lambda \in \mathbb{C} : |arg(\lambda - b)| \leq \pi - \phi\}$, with $\phi = \frac{\pi}{4}$ and $b = -\lambda_{m+1}^0 + \nu$. That is,

$$\Gamma_m = \Gamma_m^1 \cup \Gamma_m^2 = \{b + re^{-i(\pi-\phi)} : -\infty < r \leq 0\} \cup \{b + re^{i(\pi-\phi)} : 0 \leq r < +\infty\},$$

oriented such that the imaginary part grows as λ runs in Γ.

With this,

$$e^{-A_\varepsilon t}\mathbf{Q_m^\varepsilon} E - Ee^{-A_0 t}\mathbf{Q_m^0} = \frac{1}{2\pi i}\int_{\Gamma_m} \left((\lambda + A_\varepsilon)^{-1}E - E(\lambda + A_0)^{-1}\right) e^{\lambda t} d\lambda.$$

Then,

$$\|e^{-A_\varepsilon t}\mathbf{Q_m^\varepsilon} E - Ee^{-A_0 t}\mathbf{Q_m^0}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq \frac{1}{2\pi}\left|\int_{\Gamma_m} \|\left((\lambda + A_\varepsilon)^{-1}E - E(\lambda + A_0)^{-1}\right)\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} |e^{\lambda t}| d\lambda\right|,$$

applying Lemma 3.4

$$\|e^{-A_\varepsilon t}\mathbf{Q_m^\varepsilon} E - Ee^{-A_0 t}\mathbf{Q_m^0}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq \frac{\sup_{\lambda\in\Gamma_m} C_3(\lambda)\tau(\varepsilon)}{2\pi}\left|\int_{\Gamma_m} |e^{\lambda t}| d\lambda\right| = \frac{\sup_{\lambda\in\Gamma_m} C_3(\lambda)\tau(\varepsilon)}{\pi}\left|\int_{\Gamma_m^2} |e^{\lambda t}| d\lambda\right|.$$

Since $\lambda \in \Gamma_m^2$,

$$|e^{\lambda t}| = e^{(b - rcos(\phi))t}.$$

So,

$$\|e^{-A_\varepsilon t}\mathbf{Q_m^\varepsilon} E - Ee^{-A_0 t}\mathbf{Q_m^0}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq \frac{\sup_{\lambda\in\Gamma_m} C_3(\lambda)\tau(\varepsilon)}{\pi}\int_0^\infty e^{(b - rcos(\phi))t}|e^{-i(\pi-\phi)}| dr.$$

We make the change of variables $(-b + rcos(\phi))t = z$,

$$\|e^{-A_\varepsilon t}\mathbf{Q_m^\varepsilon} E - Ee^{-A_0 t}\mathbf{Q_m^0}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq \frac{\sup_{\lambda\in\Gamma_m} C_3(\lambda)\tau(\varepsilon)}{\pi cos(\phi) t}\int_{-bt}^\infty e^{-z} dz =$$

$$= \frac{\sup_{\lambda\in\Gamma_m} C_3(\lambda)}{\pi cos(\phi)} t^{-1} e^{(-\lambda_{m+1}^0 + \nu)t}\tau(\varepsilon) < 3t^{-1}e^{(-\lambda_{m+1}^0 + \nu)t}\tau(\varepsilon),$$

the last inequality is obtained taking $\phi = \frac{\pi}{4}$.

On the other side, we know that,

$$\|e^{-A_\varepsilon t}\mathbf{Q_m^\varepsilon} E - Ee^{-A_0 t}\mathbf{Q_m^0}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq$$

$$\|e^{-A_\varepsilon t}\mathbf{Q_m^\varepsilon} E\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} + \|Ee^{-A_0 t}\mathbf{Q_m^0}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)}.$$

Then, by (5.1),

$$\leq 2e^{-\lambda_{m+1}^\varepsilon t}\left(\max\{\lambda_{m+1}^\varepsilon, \frac{\alpha}{t}\}\right)^\alpha + 2e^{-\lambda_{m+1}^0 t}\left(\max\{\lambda_{m+1}^0, \frac{\alpha}{t}\}\right)^\alpha \leq$$

$$\leq 4e^{-\lambda_{m+1}^0 t}\left(\max\{(\lambda_{m+1}^0)^\alpha, t^{-\alpha}\}\right).$$

So, if we put everything together,

$$\|e^{-A_\varepsilon t}\mathbf{Q_m^\varepsilon} E - Ee^{-A_0 t}\mathbf{Q_m^0}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq 3\min\left\{t^{-1}\tau(\varepsilon), \max\{(\lambda_{m+1}^0)^\alpha, t^{-\alpha}\}\right\} e^{(-\lambda_{m+1}^0 + \nu)t} =$$

$$= 3\min\left\{t^{-1}\tau(\varepsilon), t^{-\alpha}\right\} e^{(-\lambda_{m+1}^0 + \nu)t} = 3l_\varepsilon^\alpha e^{(-\lambda_{m+1}^0 + \nu)t},$$

as we wanted to prove.

∎

We may show now the following result.

Lemma 5.4. *Let $w_\varepsilon \in \mathbf{P_m^\varepsilon} X_\varepsilon$ and $w_0 \in \mathbf{P_m^0} X_0$. Then, for ε small enough and for $0 \leq \alpha < 1$,*

$$|j_\varepsilon(w_\varepsilon) - j_0(w_0)|_\alpha \leq 3\|w_\varepsilon - Ew_0\|_{X_\varepsilon^\alpha} + 3C_P\tau(\varepsilon)\|w_0\|_{X_0}.$$

Proof. Since $\varphi_i^0=\mathbf{P}^{\mathbf{0}}_{\mathbf{m}}(\varphi_i^0)$, then if we denote by $j_\varepsilon(w_\varepsilon)=\bar p_\varepsilon$ and $j_0(w_0)=\bar p_0$

$$w_\varepsilon-Ew_0=\sum_{I=1}^m p_i^\varepsilon \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(E\varphi_i^0)-E\sum_{i=1}^m p_i^0\mathbf{P}^{\mathbf{0}}_{\mathbf{m}}\varphi_i^0=(\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}E-E\mathbf{P}^{\mathbf{0}}_{\mathbf{m}})\left(\sum_{I=1}^m p_i^\varepsilon\varphi_i^0\right)+E\sum_{i=1}^m(p_i^\varepsilon-p_i^0)\varphi_i^0$$

Applying the operator M and using that $M\circ E=I$, we get

$$\sum_{i=1}^m(p_i^\varepsilon-p_i^0)\varphi_i^0=M(w_\varepsilon-Ew_0)-M(\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}E-E\mathbf{P}^{\mathbf{0}}_{\mathbf{m}})\left(\sum_{I=1}^m p_i^\varepsilon\varphi_i^0\right)$$

Taking the X_0^α norm in the last expression and with (2.3), Lemma 3.7 and (2.12), we get

$$|\bar p_\varepsilon-\bar p_0|_\alpha\le 2\|w_\varepsilon-Ew_0\|_{X_\varepsilon^\alpha}+2C_p\tau(\varepsilon)|\bar p_\varepsilon|\le 2\|w_\varepsilon-Ew_0\|_{X_\varepsilon^\alpha}+2C_p\tau(\varepsilon)|\bar p_\varepsilon-\bar p_0|+2C_p\tau(\varepsilon)|\bar p_0|.$$

From here, we get

$$|\bar p_\varepsilon-\bar p_0|_\alpha\le\frac{2}{1-2C_P\tau(\varepsilon)}\|w_\varepsilon-Ew_0\|_{X_\varepsilon^\alpha}+\frac{2}{1-2C_P\tau(\varepsilon)}C_p\tau(\varepsilon)|\bar p_0|.$$

Taking ε small enough so that $\frac{2}{1-2C_P\tau(\varepsilon)}\le 3$ and since $|\bar p_0|=\|w_0\|_{X_0}$, we prove the result. ■

Next, we introduce some technical results.

Lemma 5.5. *For every $\Phi_\varepsilon\in\mathcal{F}_L$ with $L\le 1$ and any $\bar p^0\in\mathbb{R}^m$, if $p_\varepsilon(t)$ is the solution of (4.5), we have,*

$$\|p_\varepsilon(t)\|_{X_\varepsilon^\alpha}\le\left(\|j_\varepsilon^{-1}(\bar p^0)\|_{X_\varepsilon^\alpha}+\frac{C_F}{(\lambda_m^\varepsilon)^{1-\alpha}}\right)e^{-\lambda_m^\varepsilon t},\qquad t\le 0,$$

Proof. By the variation of constant formula for $t\le 0$,

$$\begin{aligned}\|p_\varepsilon(t)\|_{X_\varepsilon^\alpha}&\le\|e^{-A_\varepsilon t}j_\varepsilon^{-1}(\bar p^0)\|_{X_\varepsilon^\alpha}+\int_t^0\|e^{-A_\varepsilon(t-s)}\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}F_\varepsilon(p_\varepsilon(s)+\Phi_\varepsilon j_\varepsilon^{-1}(p_\varepsilon(s)))\|_{X_\varepsilon^\alpha}ds\\
&\le e^{-\lambda_m^\varepsilon t}\|j_\varepsilon^{-1}(\bar p^0)\|_{X_\varepsilon^\alpha}+\int_t^0 e^{-\lambda_m^\varepsilon(t-s)}(\lambda_m^\varepsilon)^\alpha\|\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}F_\varepsilon(p_\varepsilon(s)+\Phi_\varepsilon j_\varepsilon^{-1}(p_\varepsilon(s)))\|_{X_\varepsilon}ds\\
&\le e^{-\lambda_m^\varepsilon t}\|j_\varepsilon^{-1}(\bar p^0)\|_{X_\varepsilon^\alpha}+\int_t^0 e^{-\lambda_m^\varepsilon(t-s)}(\lambda_m^\varepsilon)^\alpha C_F ds\le\left(\|j_\varepsilon^{-1}(\bar p^0)\|_{X_\varepsilon^\alpha}+\frac{C_F}{(\lambda_m^\varepsilon)^{1-\alpha}}\right)e^{-\lambda_m^\varepsilon t}\end{aligned}$$

■

Let Φ_0 and Φ_ε be the inertial manifolds constructed above. If $\bar p^0\in\mathbb{R}^m$, we denote by $p_0(t)\in\mathbf{P}^{\mathbf{0}}_{\mathbf{m}}X_0^\alpha$ and $p_\varepsilon(t)\in\mathbf{P}^\varepsilon_{\mathbf{m}}(X_\varepsilon^\alpha)$ the solutions of the initial value problems, respectively,

$$p_{0t}=-A_0p_0+\mathbf{P}^{\mathbf{0}}_{\mathbf{m}}F_0(p_0+\Phi_0(j_0(p_0))),\qquad p_0(0)=j_0^{-1}\bar p^0,\tag{5.2}$$

and

$$p_{\varepsilon t}=-A_\varepsilon p_\varepsilon+\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}F_\varepsilon(p_\varepsilon+\Phi_\varepsilon(j_\varepsilon(p_\varepsilon))),\qquad p_\varepsilon(0)=j_\varepsilon^{-1}\bar p^0\tag{5.3}$$

We have now,

Lemma 5.6. *With the notations above, we have, for $t\le 0$,*

$$\|p_\varepsilon(t)-Ep_0(t)\|_{X_\varepsilon^\alpha}\le\left(\frac{1}{12}\sup_{\bar p\in\mathbb{R}^m}\|\Phi_\varepsilon(\bar p)-E\Phi_0(\bar p)\|_{X_\varepsilon^\alpha}+\rho(\varepsilon)+K_2(|t|+e^{-2\nu t})\tau(\varepsilon)\right)e^{-(\lambda_m^\varepsilon+4L_F(\lambda_m^\varepsilon)^\alpha)t}$$

with $K_2=(6(\lambda_m^0)^\alpha L_FC_P+C_4)(|\bar p^0|+C_F)$ and C_4 is the constant from Lemma 5.1.

Proof. To simplify the notation below, we denote by $\tilde{F}_\varepsilon = F_\varepsilon(p_\varepsilon(s) + \Phi_\varepsilon(j_\varepsilon(p_\varepsilon(s))))$ and similarly, $\tilde{F}_0 = F_0(p_0(s) + \Phi_0(j_0(p_0(s))))$. By the variation of constants formula applied to (5.2) and (5.3) we get

$$p_\varepsilon(t) - Ep_0(t) = e^{-A_\varepsilon t} j_\varepsilon^{-1}(\bar{p}^0) - Ee^{-A_0 t} j_0^{-1}(\bar{p}^0) + \int_0^t \left(e^{-A_\varepsilon (t-s)} \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} \tilde{F}_\varepsilon - Ee^{-A_0(t-s)} \mathbf{P}_{\mathbf{m}}^{\mathbf{0}} \tilde{F}_0 \right) ds$$

$$= e^{-A_\varepsilon t} j_\varepsilon^{-1}(\bar{p}^0) - Ee^{-A_0 t} j_0^{-1}(\bar{p}^0) + \int_0^t e^{-A_\varepsilon (t-s)} \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} (\tilde{F}_\varepsilon - E\tilde{F}_0) ds + \int_0^t (e^{-A_\varepsilon (t-s)} \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E - Ee^{-A_0(t-s)} \mathbf{P}_{\mathbf{m}}^{\mathbf{0}}) \tilde{F}_0 ds$$

$$= I_1 + I_2 + I_3$$

Observe that, with the definition of j_ε and with the aid of Lemma 5.1, we get

$$\|I_1\|_{X_\varepsilon^\alpha} = \|(e^{-A_\varepsilon t} \mathbf{P}_{\mathbf{m}}^\varepsilon E - Ee^{-A_0 t} \mathbf{P}_{\mathbf{m}}^{\mathbf{0}})(\sum_{i=1}^m p_i^0 \varphi_i^0)\|_{X_\varepsilon^\alpha} \leq C_4 e^{-(\lambda_m^0 + \nu)t} \tau(\varepsilon) |\bar{p}^0|$$

Moreover, we have

$$\begin{aligned} \tilde{F}_\varepsilon - E\tilde{F}_0 = {} & F_\varepsilon(p_\varepsilon + \Phi_\varepsilon(j_\varepsilon(p_\varepsilon))) - F_\varepsilon(Ep_0 + \Phi_\varepsilon(j_\varepsilon(p_\varepsilon))) \\ & + F_\varepsilon(Ep_0 + \Phi_\varepsilon(j_\varepsilon(p_\varepsilon))) - F_\varepsilon(Ep_0 + \Phi_\varepsilon(j_0(p_0))) \\ & + F_\varepsilon(Ep_0 + \Phi_\varepsilon(j_0(p_0)) - F_\varepsilon(Ep_0 + E\Phi_0(j_0(p_0))) \\ & + F_\varepsilon(Ep_0 + E\Phi_0(j_0(p_0)) - EF_0(p_0 + \Phi_0(j_0(p_0))) \end{aligned} \tag{5.4}$$

which implies

$$\|\tilde{F}_\varepsilon - E\tilde{F}_0\|_{X_\varepsilon} \leq L_F \|p_\varepsilon - Ep_0\|_{X_\varepsilon^\alpha} + L_F \cdot L |j_\varepsilon(p_\varepsilon) - j_0(p_0))|_\alpha + L_F \sup_{\bar{p} \in \mathbb{R}^m} \|\Phi_\varepsilon(\bar{p}) - E\Phi_0(\bar{p})\|_{X_\varepsilon^\alpha} + \rho(\varepsilon)$$

Taking into account Lemma 5.4 , we get

$$\|\tilde{F}_\varepsilon - E\tilde{F}_0\|_{X_\varepsilon} \leq 4L_F \|p_\varepsilon - Ep_0\|_{X_\varepsilon^\alpha} + 3L_F C_P \tau(\varepsilon) \|p_0\|_{X_0} + L_F \sup_{\bar{p} \in \mathbb{R}^m} \|\Phi_\varepsilon(\bar{p}) - E\Phi_0(\bar{p})\|_{X_\varepsilon^\alpha} + \rho(\varepsilon)$$

which implies with Lemma 5.5 and using that $\lambda_m^\varepsilon \geq 1$,

$$\begin{aligned} \|\tilde{F}_\varepsilon - E\tilde{F}_0\|_{X_\varepsilon} \leq {} & 4L_F \|p_\varepsilon - Ep_0\|_{X_\varepsilon^\alpha} + 3L_F C_P \tau(\varepsilon)(|\bar{p}^0| + C_F) e^{-\lambda_m^\varepsilon s} + \\ & + L_F \sup_{\bar{p} \in \mathbb{R}^m} \|\Phi_\varepsilon(\bar{p}) - E\Phi_0(\bar{p})\|_{X_\varepsilon^\alpha} + \rho(\varepsilon) \end{aligned} \tag{5.5}$$

In particular, we obtain:

$$\|I_2\|_{X_\varepsilon^\alpha} \leq (\lambda_m^\varepsilon)^\alpha \int_t^0 e^{-\lambda_m^\varepsilon (t-s)} \|\tilde{F}_\varepsilon - E\tilde{F}_0\|_{X_\varepsilon} ds$$

That is,

$$\begin{aligned}
\|I_2\|_{X^\alpha_\varepsilon} &\le 4L_F(\lambda^\varepsilon_m)^\alpha \int_t^0 e^{-\lambda^\varepsilon_m(t-s)}\|p_\varepsilon(s)-Ep_0(s)\|_{X^\alpha_\varepsilon}ds + (\lambda^\varepsilon_m)^\alpha 3L_F C_P(|\bar p^0|+C_F)|t|\tau(\varepsilon)e^{-\lambda^\varepsilon_m t}+\\
&\quad +(\lambda^\varepsilon_m)^\alpha\left(L_F\sup_{\bar p\in\mathbb{R}^m}\|\Phi_\varepsilon(\bar p)-E\Phi_0(\bar p)\|_{X^\alpha_\varepsilon}+\rho(\varepsilon)\right)\frac{e^{-\lambda^\varepsilon_m t}-1}{\lambda^\varepsilon_m}\\
&\le\left(\frac{L_F}{(\lambda^\varepsilon_m)^{1-\alpha}}\sup_{\bar p\in\mathbb{R}^m}\|\Phi_\varepsilon(\bar p)-E\Phi_0(\bar p)\|_{X^\alpha_\varepsilon}+\rho(\varepsilon)+K_1|t|\tau(\varepsilon)\right)e^{-\lambda^\varepsilon_m t}+\\
&\quad +4L_F(\lambda^\varepsilon_m)^\alpha\int_t^0 e^{-\lambda^\varepsilon_m(t-s)}\|p_\varepsilon(s)-Ep_0(s)\|_{X^\alpha_\varepsilon}ds\\
&\le\left(\frac{1}{12}\sup_{\bar p\in\mathbb{R}^m}\|\Phi_\varepsilon(\bar p)-E\Phi_0(\bar p)\|_{X^\alpha_\varepsilon}+\rho(\varepsilon)+K_1|t|\tau(\varepsilon)\right)e^{-\lambda^\varepsilon_m t}+\\
&\quad +4L_F(\lambda^\varepsilon_m)^\alpha\int_t^0 e^{-\lambda^\varepsilon_m(t-s)}\|p_\varepsilon(s)-Ep_0(s)\|_{X^\alpha_\varepsilon}ds
\end{aligned}$$

where we have denoted by $K_1=6(\lambda^0_m)^\alpha L_F C_P(|\bar p^0|+C_F)$ and we have used that $\lambda^\varepsilon_m>1$ and $(\lambda^\varepsilon_m)^\alpha\le 2(\lambda^0_m)^\alpha$

Finally,

$$\|I_3\|_{X^\alpha_\varepsilon}\le C_4\tau(\varepsilon)C_F\int_t^0 e^{-(\lambda^0_m+\nu)(t-s)}ds\le C_4\tau(\varepsilon)C_F e^{-(\lambda^0_m+\nu)t}$$

Putting the three expressions together, we get

$$\begin{aligned}
&\|p_\varepsilon(t)-Ep_0(t)\|_{X^\alpha_\varepsilon}\le C_4(|\bar p^0|+C_F)e^{-(\lambda^0_m+\nu)t}\tau(\varepsilon)+\\
&\left(\frac{1}{12}\sup_{\bar p\in\mathbb{R}^m}\|\Phi_\varepsilon(\bar p)-E\Phi_0(\bar p)\|_{X^\alpha_\varepsilon}+\rho(\varepsilon)+K_1|t|\tau(\varepsilon)\right)e^{-\lambda^\varepsilon_m t}+4L_F(\lambda^\varepsilon_m)^\alpha\int_t^0 e^{-\lambda^\varepsilon_m(t-s)}\|p_\varepsilon(s)-Ep_0(s)\|_{X^\alpha_\varepsilon}ds.
\end{aligned}$$

Multiplying this inequality by $e^{\lambda^\varepsilon_m t}$, denoting by $h(t)=e^{\lambda^\varepsilon_m t}\|p_\varepsilon(t)-Ep_0(t)\|_{X^\alpha_\varepsilon}$ and assuming ε is small enough so that $|\lambda^\varepsilon_m-\lambda^0_m|<\nu$, we may write

$$h(t)\le\left(\frac{1}{12}\sup_{\bar p\in\mathbb{R}^m}\|\Phi_\varepsilon(\bar p)-E\Phi_0(\bar p)\|_{X^\alpha_\varepsilon}+\rho(\varepsilon)+K_2(|t|+e^{-2\nu t})\tau(\varepsilon)\right)+4L_F(\lambda^\varepsilon_m)^\alpha\int_t^0 h(s)ds$$

where $K_2=(6(\lambda^0_m)^\alpha L_F C_P+C_4)(|\bar p^0|+C_F)$. Applying Gronwall inequality, we get,

$$h(t)\le\left(\frac{1}{12}\sup_{\bar p\in\mathbb{R}^m}\|\Phi_\varepsilon(\bar p)-E\Phi_0(\bar p)\|_{X^\alpha_\varepsilon}+\rho(\varepsilon)+K_2(|t|+e^{-2\nu t})\tau(\varepsilon)\right)e^{-4L_F(\lambda^\varepsilon_m)^\alpha t}$$

which implies that

$$\|p_\varepsilon(t)-Ep_0(t)\|_{X^\alpha_\varepsilon}\le\left(\frac{1}{12}\sup_{\bar p\in\mathbb{R}^m}\|\Phi_\varepsilon(\bar p)-E\Phi_0(\bar p)\|_{X^\alpha_\varepsilon}+\rho(\varepsilon)+K_2(|t|+e^{-2\nu t})\tau(\varepsilon)\right)e^{-(\lambda^\varepsilon_m+4L_F(\lambda^\varepsilon_m)^\alpha)t}$$

which shows the result. ■

With these results, we have all the needed tools to estimate the rate of convergence of the inertial manifolds, proving the main result of the article

Proof. Notice that we have

$$\Phi_0(\bar{p}^0)=\int_{-\infty}^{0} e^{A_0 s}\mathbf{Q}_m^0 F_0\big(p_0(s)+\Phi_0(j_0(p_0(s)))\big)ds, \tag{5.6}$$

and

$$\Phi_\varepsilon(\bar{p}^0)=\int_{-\infty}^{0} e^{A_\varepsilon s}\mathbf{Q}_m^\varepsilon F_\varepsilon\big(p_\varepsilon(s)+\Phi_\varepsilon(j_\varepsilon(p_\varepsilon(s)))\big)ds, \tag{5.7}$$

where $p_0(s)$ and $p_\varepsilon(s)$ are the solutions of (5.2) and (5.3). Denoting, as in the proof of the previous Lemma, $\tilde{F}_\varepsilon=F_\varepsilon(p_\varepsilon(s)+\Phi_\varepsilon(j_\varepsilon(p_\varepsilon(s))))$ and $\tilde{F}_0=F_0(p_0(s)+\Phi_0(j_0(p_0(s))))$

$$\Phi_\varepsilon(\bar{p}^0)-E\Phi_0(\bar{p}^0)=\int_{-\infty}^{0}\left(e^{A_\varepsilon s}\mathbf{Q}_m^\varepsilon \tilde{F}_\varepsilon - Ee^{A_0 s}\mathbf{Q}_m^0\tilde{F}_0\right)ds=$$

$$=\int_{-\infty}^{0} e^{A_\varepsilon s}\mathbf{Q}_m^\varepsilon(\tilde{F}_\varepsilon-E\tilde{F}_0)ds+\int_{-\infty}^{0}\left(e^{A_\varepsilon s}\mathbf{Q}_m^\varepsilon E-Ee^{A_0 s}\mathbf{Q}_m^0\right)\tilde{F}_0 ds=I_1+I_2.$$

With (5.1)

$$\|I_1\|_{X_\varepsilon^\alpha}\le\int_{-\infty}^{0} e^{\lambda_{m+1}^\varepsilon s}\left(\max\{\lambda_{m+1}^\varepsilon,\frac{\alpha}{t}\}\right)^\alpha\|\tilde{F}_\varepsilon-E\tilde{F}_0\|_{X_\varepsilon}ds.$$

Now, with the decomposition as in (5.4) and with (5.5) and denoting by $\|E\Phi_0-\Phi_\varepsilon\|_\infty=\|E\Phi_0-\Phi_\varepsilon\|_{L^\infty(\mathbb{R}^m,X_\varepsilon^\alpha)}$, we obtain

$$\|I_1\|_{X_\varepsilon^\alpha}\le\int_{-\infty}^{0} e^{\lambda_{m+1}^\varepsilon s}\left(\max\{\lambda_{m+1}^\varepsilon,\frac{\alpha}{s}\}\right)^\alpha\Big[4L_F\|p_\varepsilon(s)-Ep_0(s)\|_{X_\varepsilon^\alpha}+3L_F C_p\tau(\varepsilon)(|\bar{p}^0|+C_F)e^{-\lambda_m^\varepsilon s}+$$
$$+L_F\|E\Phi_0-\Phi_\varepsilon\|_\infty+\rho(\varepsilon)\Big]ds$$

$$=4L_F\int_{-\infty}^{0} e^{\lambda_{m+1}^\varepsilon s}\left(\max\{\lambda_{m+1}^\varepsilon,\frac{\alpha}{s}\}\right)^\alpha\|p_\varepsilon(s)-Ep_0(s)\|_{X_\varepsilon^\alpha}ds+$$
$$+3L_F C_p\tau(\varepsilon)(|\bar{p}^0|+C_F)\int_{-\infty}^{0} e^{(\lambda_{m+1}^\varepsilon-\lambda_m^\varepsilon)s}\left(\max\{\lambda_{m+1}^\varepsilon,\frac{\alpha}{s}\}\right)^\alpha ds+$$
$$+\rho(\varepsilon)\int_{-\infty}^{0} e^{\lambda_{m+1}^\varepsilon s}\left(\max\{\lambda_{m+1}^\varepsilon,\frac{\alpha}{s}\}\right)^\alpha ds$$
$$+L_F\|E\Phi_0-\Phi_\varepsilon\|_\infty\int_{-\infty}^{0} e^{\lambda_{m+1}^\varepsilon s}\left(\max\{\lambda_{m+1}^\varepsilon,\frac{\alpha}{s}\}\right)^\alpha ds.$$

The second term in the last expression can be estimated with Lemma 5.2, since

$$\int_{-\infty}^{0} e^{(\lambda_{m+1}^\varepsilon-\lambda_m^\varepsilon)s}\left(\max\{\lambda_{m+1}^\varepsilon,\frac{\alpha}{s}\}\right)^\alpha\le(1-\alpha)^{-1}(\lambda_{m+1}^\varepsilon)^{\alpha-1}+(\lambda_{m+1}^\varepsilon)^\alpha(\lambda_{m+1}^\varepsilon-\lambda_m^\varepsilon)^{-1}$$

which is uniformly bounded as $\varepsilon\to0$. Then, the second term is bounded by $C(|\bar{p}^0|+1)\tau(\varepsilon)$ with C a constant independent of ε. Similar estimate is obtained for the third term: it will be bounded by $C\rho(\varepsilon)$ with C a constant independent of ε.

For the fourth term

$$\int_{-\infty}^{0} e^{\lambda_{m+1}^\varepsilon s}\left(\max\{\lambda_{m+1}^\varepsilon,\frac{\alpha}{s}\}\right)^\alpha\le(1-\alpha)^{-1}(\lambda_{m+1}^\varepsilon)^{\alpha-1}+(\lambda_{m+1}^\varepsilon)^{\alpha-1}\le2(1-\alpha)^{-1}(\lambda_{m+1}^\varepsilon)^{\alpha-1}.$$

Which implies that ,

$$L_F\|E\Phi_0-\Phi_\varepsilon\|_\infty\int_{-\infty}^{0} e^{\lambda_{m+1}^\varepsilon s}\left(\max\{\lambda_{m+1}^\varepsilon,\frac{\alpha}{s}\}\right)^\alpha ds\le2L_F(1-\alpha)^{-1}(\lambda_{m+1}^\varepsilon)^{\alpha-1}\|E\Phi_0-\Phi_\varepsilon\|_\infty$$

The first term need to be estimated with the aid of Lemma 5.6. Notice that,

$$4L_F\int_{-\infty}^0 e^{\lambda_{m+1}^\varepsilon s}\Big(\max\{\lambda_{m+1}^\varepsilon,\frac{\alpha}{s}\}\Big)^\alpha \|p_\varepsilon(s)-Ep_0(s)\|_{X_\varepsilon^\alpha}ds\le$$

$$\le \frac{L_F}{3}\|E\Phi_0-\Phi_\varepsilon\|_\infty\int_{-\infty}^0 e^{(\lambda_{m+1}^\varepsilon-\lambda_m^\varepsilon-4L_F(\lambda_m^\varepsilon)^\alpha)s}\Big(\max\{\lambda_{m+1}^\varepsilon,\frac{\alpha}{s}\}\Big)^\alpha ds+$$

$$+4L_F\rho(\varepsilon)\int_{-\infty}^0 e^{(\lambda_{m+1}^\varepsilon-\lambda_m^\varepsilon-4L_F(\lambda_m^\varepsilon)^\alpha)s}\Big(\max\{\lambda_{m+1}^\varepsilon,\frac{\alpha}{s}\}\Big)^\alpha ds+$$

$$+4K_2L_F\tau(\varepsilon)\int_{-\infty}^0 e^{(\lambda_{m+1}^\varepsilon-\lambda_m^\varepsilon-4L_F(\lambda_m^\varepsilon)^\alpha)s}\Big(\max\{\lambda_{m+1}^\varepsilon,\frac{\alpha}{s}\}\Big)^\alpha (|s|+e^{-2\nu s})ds$$

With similar arguments as above, the last two terms are bounded by $C\rho(\varepsilon)$ and $C\tau(\varepsilon)$ with C a constant independent of ε.

The first term is bounded by

$$\frac{L_F}{3}\|E\Phi_0-\Phi_\varepsilon\|_\infty\left((1-\alpha)^{-1}(\lambda_{m+1}^\varepsilon)^{\alpha-1}+\frac{(\lambda_{m+1}^\varepsilon)^\alpha}{\lambda_{m+1}^\varepsilon-\lambda_m^\varepsilon-4L_F(\lambda_m^\varepsilon)^\alpha}\right)$$

Putting all these estimates together, we have

$$\|I_1\|_{X_\varepsilon^\alpha}\le\left[2L_F(1-\alpha)^{-1}(\lambda_{m+1}^\varepsilon)^{\alpha-1}+\frac{L_F}{3}\left((1-\alpha)^{-1}(\lambda_{m+1}^\varepsilon)^{\alpha-1}+\frac{(\lambda_{m+1}^\varepsilon)^\alpha}{\lambda_{m+1}^\varepsilon-\lambda_m^\varepsilon-4L_F(\lambda_m^\varepsilon)^\alpha}\right)\right]\|E\Phi_0-\Phi_\varepsilon\|_\infty+$$

$$+C(|\bar p^0|+1)\tau(\varepsilon)+C\rho(\varepsilon)\le\left(3L_F(1-\alpha)^{-1}(\lambda_{m+1}^\varepsilon)^{\alpha-1}+\frac{L_F(\lambda_{m+1}^\varepsilon)^\alpha}{\lambda_{m+1}^\varepsilon-\lambda_m^\varepsilon-4L_F(\lambda_m^\varepsilon)^\alpha}\right)\|E\Phi_0-\Phi_\varepsilon\|_\infty$$

$$+C(|\bar p^0|+1)\tau(\varepsilon)+C\rho(\varepsilon)\le\frac12\|E\Phi_0-\Phi_\varepsilon\|_\infty+C(|\bar p^0|+1)\tau(\varepsilon)+C\rho(\varepsilon)$$

where we have used (4.3).

Now we estimate I_2.

$$\|I_2\|_{X_\varepsilon^\alpha}\le\int_{-\infty}^0\|\left(e^{A_\varepsilon s}\mathbf{Q}_m^\varepsilon-Ee^{A_0s}\mathbf{Q}_m^0\right)\|_{\mathcal{L}(X_0,X_\varepsilon^\alpha)}\|\tilde F_0\|_{X_0}ds\le\int_{-\infty}^0 3e^{-(\lambda_{m+1}^0-\nu)t}l_\varepsilon^\alpha(t)C_Fdt\le\frac{6C_F}{1-\alpha}\tau(\varepsilon)|\log(\tau(\varepsilon))|$$

where we have used Lemma 5.3 and Lemma 3.10.

Putting together the estimates for I_1 and I_2, we get

$$\|\Phi_\varepsilon(\bar p^0)-E\Phi_0(\bar p^0)\|_{X_\varepsilon^\alpha}\le\frac12\|\Phi_\varepsilon-E\Phi_0\|_\infty+C(|\bar p^0|+1)\tau(\varepsilon)+C\rho(\varepsilon)+\frac{6C_F}{1-\alpha}\tau(\varepsilon)|\log(\tau(\varepsilon))|$$

Now since Φ_ε and Φ_0 are of compact support, we take the sup norm for $\bar p^0$ with $|\bar p^0|\le R$, where R is an upper bound of the support of all inertial manifolds and obtain

$$\|\Phi_\varepsilon-E\Phi_0\|_\infty\le\frac12\|E\Phi_0-\Phi_\varepsilon\|_\infty+C(R+1)\tau(\varepsilon)+C\rho(\varepsilon)+\frac{6C_F}{1-\alpha}\tau(\varepsilon)|\log(\tau(\varepsilon))|$$

which implies that

$$\|\Phi_\varepsilon-E\Phi_0\|_\infty\le C(\rho(\varepsilon)+\tau(\varepsilon)|\log(\tau(\varepsilon))|)$$

which shows the theorem. ■

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